INTEGRABLE AND CHAOTIC MOTIONS OF FOUR VORTICES

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Conclusive numerical evidence of chaos in the four-vortex problem is presented using the method of Poincaré sections. The problem is formally reduced to a two degrees of freedom hamiltonian. The advection of a passive marker by three vortices displays chaos.

The hamiltonian dynamics of a system of N point vortices [1] has the remarkable property of being integrable for $N = 3$ [2,3]. Recent numerical experiments by Novikov and Sedov [4,5] (see also ref. [6]) suggest that already a system of four identical vortices will display chaotic motions. In this letter we report on similar experiments, at considerably higher numerical resolution, that show conclusively the onset of large scale chaos.

We first describe an intuitive way of producing "Poincaré sections" for the motion of four identical vortices. Labelling the vortices 1–4 we focus on their relative positions in the 8-dimensional configuration space of points (x, y). It is well known that the same coordinates define the system phase space [1]. As basic variables we consider six-tuples of euclidean separations $(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34})$. There is a geometrical constraint on these six separations; only five of them are independent. There are furthermore two dynamical constraints, one coming from the conservation of angular momentum,

$$d_{12}^2 + d_{13}^2 + d_{14}^2 + d_{23}^2 + d_{24}^2 + d_{34}^2 = \text{constant}\,, \qquad (1)$$

the other,

$$d_{12}d_{13}d_{14}d_{23}d_{24}d_{34} = \text{constant}\,, \qquad (2)$$

from conservation of the kinetic energy of interaction of the vortices [1]. Hence, a system point in the space of separation six-tuples is constrained to move on some bounded three-dimensional manifold depending on the values of the isolating integrals (1) and (2). As the motion evolves we monitor the separations, and to define a section record the values of the pair (d_{23}^2, d_{31}^2) whenever $d_{12}^2 = x$ and $\dot{d}_{12} > 0$. Varying x we obtain a one-parameter family of sections. Since the vortices are identical any configuration produces by simple relabelling $4! = 24$ different initial conditions with the same values of the integrals (1) and (2).

Figs. 1 and 2 show examples of sections. Due to a discrete symmetry of the point vortex equations of motion, configurations with the four vortices at the vertices of a rectangle define periodic solutions which yield a single point in a section of this type. Such fixed points are shown as crosses. Fig. 1 was produced by perturbing (in five different ways) a rectangle of aspect ratio 1.5 and making use of the permutation symmetry. A total of approximately 1.5×10^6 time steps was necessary to produce fig. 1 resulting in a

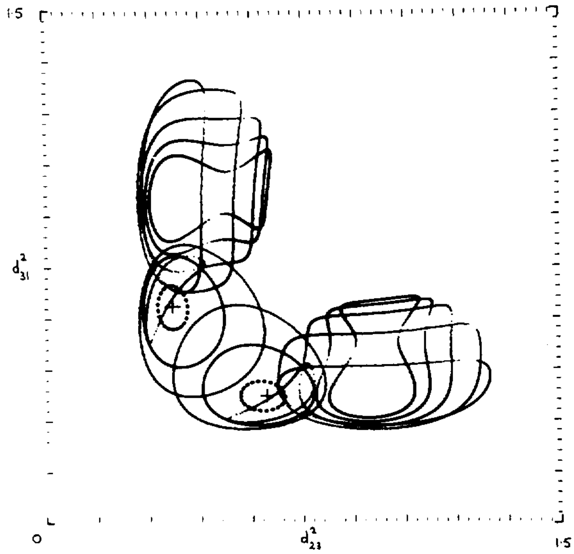

Fig. 1. Poincaré section for perturbations of aspect ratio 1.5 rectangle configuration (x = 0.75).

total of 10 332 points in the section. The integral (2) was conserved to within one part in 10^7 during integration. The other integrals, (1) and the position of the center of vorticity, were conserved to much higher accuracy since they arise from spatial symmetries of the interaction [1]. Halving the time step in the fifth order Fehlberg Runge–Kutta time-stepping routine used reproduced the plotted points to at least five digits. This is beyond the plotter accuracy.

A large number of apparently smooth curves are visible in the section of fig. 1 suggesting at least approximate integrability in the vicinity of the periodic orbit. We remark that the intersections of the curves seen in fig. 1 are not in conflict with uniqueness properties of the solution. The six separations do not uniquely specify a configuration. We also note that the result in fig. 1 is consistent with the KAM theorem [7] which in the case at hand would say that the system is predominantly integrable in the vicinity of the stable, uniformly rotating, square configuration (aspect ratio unity).

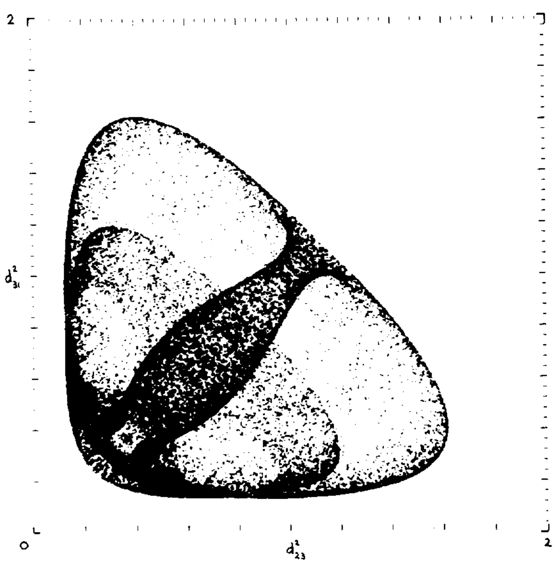

Fig. 2. Poincaré section for perturbations of centered equilateral triangle configuration (x = 0.75).

By contrast, fig. 2 shows the section resulting from a singly perturbation of the centered equilateral triangle configuration. This section contains approximately 20 000 points and represents in effect a probability density in phase space. Qualitative differences also appear in plots of the real space trajectories of the vortices. The random splatter of points seen in fig. 2 has been reproduced with shorter, time reversible runs. We may add that the Poincaré section for perturbations of an aspect ratio 2 rectangle showed similar evidence of chaos.

It is possible and desirable to reproduce the preceding results using a more systematic approach. We have used a sequence of canonical transformations to reduce the problem of four identical vortices to a two degrees of freedom hamiltonian. The key ingredient in these formal developments is the following transformation (written here for arbitrary N):

$$q_n + \mathrm{i}p_n = N^{-1/2} \sum_{\alpha=1}^{N} z_\alpha \exp[\mathrm{i}2\pi n(\alpha - 1)/N] , \quad n = 0, 1, \ldots, N-1 , \tag{3}$$

i.e. a discrete Fourier transform (DFT) of the positions, $z_\alpha = x_\alpha + \mathrm{i}y_\alpha$, considered as an array of complex data [‡1]. As anticipated by the notation the variables q_n, p_n are canonically conjugate. Further q_0, p_0 are constants of the motion and hence do not appear in the transformed hamiltonian. Introducing "action-angle" variables J_n, θ_n by

‡1 We are indebted to R. Littlejohn for pointing out this transformation.

$$(2J_n)^{1/2}\exp(i\theta_n) = q_n + ip_n\ , \tag{4}$$

we then notice, that the hamiltonian depends on the angles only through the combinations $\theta_1 - \theta_3, \theta_2 - \theta_3$. Also by Parseval's theorem applied to the DFT, eq. (3), $J_1 + J_2 + J_3 \equiv I_3$ is a constant of the motion. A final canonical transformation with generating function [8]

$$F_3(J_1, J_2, J_3, \phi_1, \phi_2, \phi_3) = \phi_1(J_1 - J_3) + \phi_2(J_1 + J_3) + \phi_3(J_1 + J_2 + J_3)\ , \tag{5}$$

then produces a reduced hamiltonian,

$$H(I_1, I_2, \phi_1, \phi_2) = -(4\pi)^{-1}\log\left[h(I_1, I_2, \phi_1, \phi_2)\right]\ , \tag{6}$$

with two degrees of freedom. By a rescaling of time we may consider the argument of the logarithm, h, to be the governing hamiltonian for given initial conditions. The explicit expression for h is cumbersome involving 14 terms of the form $f_{mn}\cos(m\phi_1 + n\phi_2)$, where m, n are integers and the coefficients are algebraic functions of I_1, I_2 with I_3 as a parameter. It is reminiscent of but more complicated than hamiltonians appearing in problems with coupled oscillators, known to produce chaotic motions due to the phenomenon of resonance overlap [9]. Numerical computations based on this representation corroborate the conclusions from figs. 1 and 2. We note in passing that by steps analogous to (3)–(6) the problem of three identical vortices may be reduced to a one degree of freedom hamiltonian, the action variable having the physical interpretation of the area of the vortex triangle. Explicit expressions in terms of Jacobi elliptic functions may then be obtained for the variation in time of the relative configuration of the vortices, providing a more detailed solution of this problem than in previous work [2,3].

Certain other four-vortex problems are potentially of greater interest than the motion of identical vortices. We have considered what may be called the "restricted" four-vortex problem, i.e. the advection of a passive marker particle (a "vortex" of strength zero) by three identical vortices. Here resonant interaction between rotation of the marker and the relative motion of the vortices leads to stochasticity. Figs. 3 and 4 show sequences of positions of the marker particle at time intervals equal to the period of relative motion of the vortices. In fig. 3 the angular velocity of the marker particle was chosen too small for resonance and the stroboscopic point plot reveals an apparently smooth curve. In fig. 4 on the other hand the distance of the marker particle from the centroid was decreased, with a consequent increase in angular velocity, and a chaotic splatter of points ensues. We note that although in the continuum limit the advec-

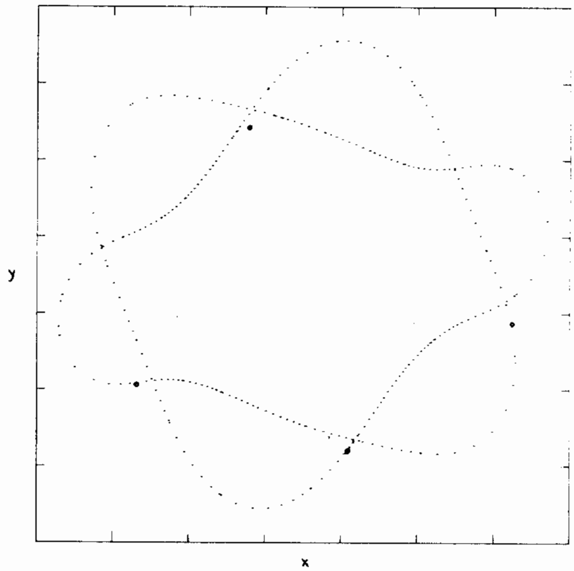

Fig. 3. Stroboscopic point plot of passive marker advected by three vortices. Initial vortex positions ○, marker position •.

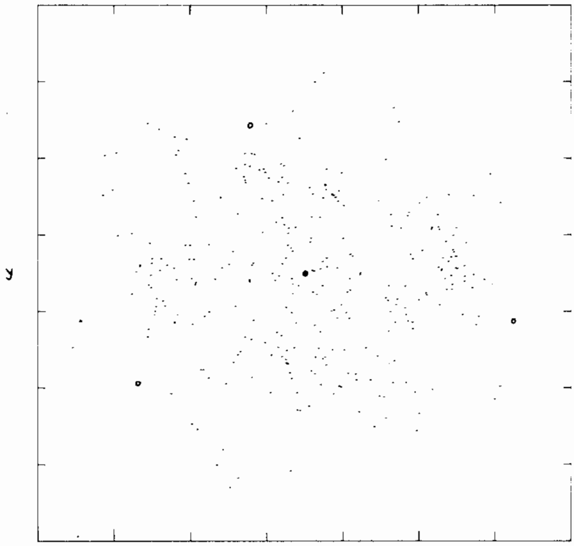

Fig. 4. Stroboscopic point plot of passive marker advected by three vortices. Initial vortex positions ○, marker position •.

tion of a passive marker is governed by a linear equation, and is thus often considered to be simpler than the flow dynamics itself, this example shows just the opposite: The motion of three vortices is integrable; the motion of a particle advected by their velocity field is non-integrable, hence infinitely more complicated.

Finally we have considered the collision of two vortex pairs. Again the discrete symmetries yield integrable solutions, e.g. coaxial pairs [10] or two pairs with opposite velocities of equal magnitude, and again we expect chaotic motions when these symmetries are broken, now in the context of a scattering problem. We speculate that one consequence of the very sensitive dependence on initial conditions is that the scattering angle may be considered a random variable, large ranges of scattering angle corresponding to small changes in impact parameter. This suggests interesting simple models of the collision term in the kinetics of a "dilute gas" of vortex pairs. A detailed account of the work reported here will appear elsewhere.

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